



IB Demystified

Examiners | Moderators | Mentors

Mathematics: Analysis and Approaches Higher Level

Paper 3 — Mock Examination

Mock Examination 2
Worked Solutions & Markscheme

Calculator: A graphic display calculator is **required**
Time allowed: 1 hour 15 minutes
Maximum mark: 55 marks

How to use this booklet

Each part gives the investigation aim, given information, the result used, a strategy, the thought process, three progressive hints, a **GDC Method** box where appropriate, full vertical working with marks, a boxed answer, and an examiner note. Codes: **M** method, **A** accuracy, **R** reasoning, **AG** answer given, **GDC M/GDC A** calculator method/output.

Question 1**[Maximum mark: 27]****Investigation Aim**

To study the square-root iteration $u_{n+1} = \frac{1}{2}\left(u_n + \frac{c}{u_n}\right)$: identify its limit, prove convergence by a monotone-bounded argument, and explain its rapid (quadratic) convergence.

Given

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{c}{u_n}\right) \text{ with } c > 0, u_1 > 0.$$

Formula / Law Used

Fixed point: a limit L satisfies $L = \frac{1}{2}\left(L + \frac{c}{L}\right)$. A sequence that is decreasing and bounded below converges (monotone convergence).

Strategy

Tabulate; solve the fixed-point equation; manipulate $u_{n+1} - \sqrt{c}$ into a perfect square; use that to prove the lower bound and the decrease; combine for convergence; analyse the error to explain digit-doubling.

Thought Process

The single algebraic identity in part (c) does almost all the work: it shows each term stays above \sqrt{c} , that the gap shrinks, and that the new gap is essentially the square of the old one.

Hint 1: For (b), replace every u in the recurrence by L and solve.

Hint 2: For (c), put $\frac{1}{2}(u_n + c/u_n) - \sqrt{c}$ over the common denominator $2u_n$.

Hint 3: For (h), bound $2u_n \geq 2\sqrt{c}$ in part (c) to get $\text{error}_{n+1} \leq \text{error}_n^2 / (2\sqrt{c})$.

(a)

With $c = 7$, $u_1 = 1$:

$$u_2 = \frac{1}{2}\left(1 + \frac{7}{1}\right) = 4.000000$$

$$u_3 = \frac{1}{2}\left(4 + \frac{7}{4}\right) = 2.875000$$

$$u_4 = \frac{1}{2}\left(2.875 + \frac{7}{2.875}\right) = 2.654891$$

$$u_5 = \frac{1}{2}\left(2.654891 + \frac{7}{2.654891}\right) = 2.645767$$

M1A1A1 The terms approach $\sqrt{7} = 2.645751 \dots$

A1

$$u_2 = 4.000000, u_3 = 2.875000, u_4 = 2.654891, u_5 = 2.645767; \rightarrow \sqrt{7}$$

GDC Method

Recurrence table. Enter the rule in sequence/recursive mode or a spreadsheet: $u_1 = 1$, $u_{n+1} = 0.5(u_n + 7/u_n)$. *TI-Nspire:* use a Lists & Spreadsheet column =0.5*(b1+7/b1) filled down. *Casio:* Recursion menu, type $a_{n+1} = 0.5(a_n + 7 \div a_n)$.

(b)

$$\begin{aligned}
 L &= \frac{1}{2} \left(L + \frac{c}{L} \right) \\
 2L &= L + \frac{c}{L} \\
 L &= \frac{c}{L} \\
 L^2 = c &\Rightarrow L = \sqrt{c} \quad (L > 0)
 \end{aligned}$$

M1A1 A1

$$L = \sqrt{c}$$

(c)

$$\begin{aligned}
 u_{n+1} - \sqrt{c} &= \frac{1}{2} \left(u_n + \frac{c}{u_n} \right) - \sqrt{c} \\
 &= \frac{u_n^2 + c - 2\sqrt{c}u_n}{2u_n} \\
 &= \frac{(u_n - \sqrt{c})^2}{2u_n}
 \end{aligned}$$

M1A1A1

$$u_{n+1} - \sqrt{c} = \frac{(u_n - \sqrt{c})^2}{2u_n}$$

AG

Examiner note. Recognising $u_n^2 - 2\sqrt{c}u_n + c$ as a perfect square is the key step. Do not expand $()^2$ back out.

(d)

The right side of (c) is a square divided by a positive number ($u_n > 0$), hence

$$u_{n+1} - \sqrt{c} = \frac{(u_n - \sqrt{c})^2}{2u_n} \geq 0 \Rightarrow u_{n+1} \geq \sqrt{c}.$$

M1A1 For $n \geq 2$ every term is produced by this rule, so $u_n \geq \sqrt{c}$ for all $n \geq 2$ (the only way equality holds is if some $u_n = \sqrt{c}$, after which the sequence is constant). **R1**

$$u_n > \sqrt{c} \text{ (or } = \sqrt{c}) \text{ for all } n \geq 2$$

(e)

For $n \geq 2$ we have $u_n \geq \sqrt{c}$, so $u_n^2 \geq c$ and

$$\begin{aligned}
 u_{n+1} - u_n &= \frac{1}{2} \left(u_n + \frac{c}{u_n} \right) - u_n \\
 &= \frac{c - u_n^2}{2u_n} \leq 0
 \end{aligned}$$

M1A1 R1

$$u_{n+1} \leq u_n \text{ for } n \geq 2 \text{ (the sequence is decreasing)}$$

(f)

From (d) the sequence is bounded below by \sqrt{c} , and from (e) it is decreasing for $n \geq 2$. A decreasing sequence that is bounded below converges. By part (b) its only possible positive limit is \sqrt{c} . **R1**

The sequence converges, with limit \sqrt{c} .

A1

Examiner note. This is the monotone convergence argument: “bounded + monotone \Rightarrow convergent”. Both ingredients must be cited.

(g)

GDC Method

Threshold search. Extend the recurrence table from part (a). Add a column $|u_n - \sqrt{7}|$ and read off the first n for which it is below 10^{-9} .

$$|u_5 - \sqrt{7}| \approx 1.57 \times 10^{-5}, \quad |u_6 - \sqrt{7}| \approx 4.68 \times 10^{-11}.$$

GDC M1 A1

Least $n = 6$

A1

(h)

From part (c), for $n \geq 2$ we have $2u_n \geq 2\sqrt{c}$, so

$$\begin{aligned} |u_{n+1} - \sqrt{c}| &= \frac{(u_n - \sqrt{c})^2}{2u_n} \\ &\leq \frac{1}{2\sqrt{c}} (u_n - \sqrt{c})^2. \end{aligned}$$

M1A1 Write the error as $e_n = |u_n - \sqrt{c}|$. Then $e_{n+1} \leq \frac{e_n^2}{2\sqrt{c}}$. **M1** If u_n is correct to d decimal places, $e_n \approx 10^{-d}$, so

$$e_{n+1} \lesssim \frac{10^{-2d}}{2\sqrt{c}}.$$

The exponent of 10 roughly doubles from d to $2d$, i.e. the number of correct decimal places approximately doubles each step. **R1**

Squaring the error each step ($e_{n+1} \lesssim e_n^2 / (2\sqrt{c})$) doubles the number of correct digits — quadratic convergence.

A1

Examiner note / generalisation. The table in (a) shows this directly: errors $\sim 10^{-1}, 10^{-3}, 10^{-5}, 10^{-11}$ — each is about the square of the one before. This is why two or three iterations already give machine precision.

Total: 27 marks (a)4 (b)3 (c)4 (d)3 (e)3 (f)2 (g)3 (h)5

How this connects to the investigation: the table suggests a limit (a); the fixed point identifies it (b); the squared-error identity (c) supplies both the bound (d) and the decrease (e); together they prove convergence (f); the GDC quantifies the speed (g); and (h) explains that speed.

Question 2

[Maximum mark: 28]

Investigation Aim

To classify two lines in space, measure the gap between them, locate the closest points, and then tune a direction parameter so the lines meet.

Given

$L_1 : \mathbf{r} = (1, 0, 2) + s(2, 1, -1)$, $L_2 : \mathbf{r} = (0, 3, 1) + t(1, -1, 2)$, and a family $L_2(k)$ with direction $(1, -1, k)$.

Formula / Law Used

Direction vectors $\mathbf{d}_1, \mathbf{d}_2$; angle $\cos \theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}$; shortest distance between skew lines $\frac{|(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$;
coplanarity (intersection) $\Leftrightarrow (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 0$.

Strategy

Try to solve for an intersection (failure \Rightarrow skew, since not parallel); use the cross product for angle and distance; impose perpendicularity for the feet; set the scalar triple product to zero for the family.

Thought Process

The vector $\mathbf{d}_1 \times \mathbf{d}_2$ is perpendicular to both lines, so it gives the angle (via the dot product), the distance (via projection of $\mathbf{b} - \mathbf{a}$), and the intersection condition (when that projection is zero).

Hint 1: For (a), set the two position vectors equal and show the system has no solution.

Hint 2: For (c), project $\mathbf{b} - \mathbf{a}$ onto $\mathbf{d}_1 \times \mathbf{d}_2$.

Hint 3: For (f), the lines are coplanar exactly when $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 0$.

(a)

Set $L_1 = L_2$:

$$1 + 2s = t \quad (1)$$

$$s = 3 - t \quad (2)$$

$$2 - s = 1 + 2t \quad (3)$$

From (1),(2): $s = 3 - (1 + 2s) \Rightarrow 3s = 2 \Rightarrow s = \frac{2}{3}$, $t = \frac{7}{3}$. **M1A1** Check (3): LHS = $2 - \frac{2}{3} = \frac{4}{3}$, RHS = $1 + \frac{14}{3} = \frac{17}{3}$; $\frac{4}{3} \neq \frac{17}{3}$, so no common point. **M1** Also $\mathbf{d}_1 = (2, 1, -1)$ is not a multiple of $\mathbf{d}_2 = (1, -1, 2)$, so the lines are not parallel. **R1**

No intersection and not parallel \Rightarrow the lines are skew.

(b)

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = (2)(1) + (1)(-1) + (-1)(2) = -1$$

$$|\mathbf{d}_1| = |\mathbf{d}_2| = \sqrt{6}$$

$$\cos \theta = \frac{|-1|}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$$

M1A1

$$\theta = \arccos \frac{1}{6} = 80.4^\circ (1.40 \text{ rad})$$

A1

(c)

$$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, \quad |\mathbf{d}_1 \times \mathbf{d}_2| = \sqrt{35}$$

$$\mathbf{b} - \mathbf{a} = (-1, 3, -1)$$

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = (-1)(1) + (3)(-5) + (-1)(-3) = -13$$

M1A1A1

$$\text{distance} = \frac{|-13|}{\sqrt{35}}$$

M1

$$d = \frac{13}{\sqrt{35}} = 2.20 (2.1974\dots)$$

A1

Examiner note. Use the *exact* cross product and surd here; rounding $\sqrt{35}$ early loses accuracy that later parts may need.

(d)

Let $P = \mathbf{a} + s\mathbf{d}_1$, $Q = \mathbf{b} + t\mathbf{d}_2$, with $\overrightarrow{PQ} = (\mathbf{b} - \mathbf{a}) + t\mathbf{d}_2 - s\mathbf{d}_1$. Require $\overrightarrow{PQ} \cdot \mathbf{d}_1 = 0$ and $\overrightarrow{PQ} \cdot \mathbf{d}_2 = 0$:

$$\overrightarrow{PQ} \cdot \mathbf{d}_1 = 0 : 6s + t = 2$$

$$\overrightarrow{PQ} \cdot \mathbf{d}_2 = 0 : s + 6t = 6$$

$$\text{M1A1 Solving: } s = \frac{6}{35}, t = \frac{34}{35}.$$

A1

$$P = \left(\frac{47}{35}, \frac{6}{35}, \frac{64}{35}\right) \approx (1.34, 0.171, 1.83)$$

$$Q = \left(\frac{34}{35}, \frac{71}{35}, \frac{103}{35}\right) \approx (0.971, 2.03, 2.94)$$

A1

$$P = \left(\frac{47}{35}, \frac{6}{35}, \frac{64}{35}\right), \quad Q = \left(\frac{34}{35}, \frac{71}{35}, \frac{103}{35}\right)$$

Examiner note. As a check, $\overrightarrow{PQ} = \frac{13}{35}(-1, 5, 3)$, whose length is $\frac{13\sqrt{35}}{35} = \frac{13}{\sqrt{35}}$, matching part (c).

(e)

GDC Method

Numerical minimisation. Define

$$D(s, t) = |(\mathbf{a} + s\mathbf{d}_1) - (\mathbf{b} + t\mathbf{d}_2)|^2,$$

a function of two variables. On the GDC either minimise \sqrt{D} over (s, t) , or substitute $s = \frac{6}{35}$, $t = \frac{34}{35}$ to evaluate. A solver/minimum tool returns the minimum near $(s, t) = (0.171, 0.971)$ with $\sqrt{D_{\min}} \approx 2.197$.

GDC M1 GDC A1

$$\sqrt{D_{\min}} = 2.197\dots = \frac{13}{\sqrt{35}}, \text{ confirming part (c).}$$

A1A1

(f)

For $L_2(k)$ with $\mathbf{d}_2 = (1, -1, k)$:

$$\begin{aligned} \mathbf{d}_1 \times \mathbf{d}_2 &= (k - 1, -(2k + 1), -3) \\ (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) &= (-1)(k - 1) + 3(-(2k + 1)) + (-1)(-3) \\ &= -7k + 1 \end{aligned}$$

M1A1 Lines meet \Leftrightarrow coplanar (and not parallel) $\Leftrightarrow -7k + 1 = 0$.

M1

$$k = \frac{1}{7}$$

A1

(g)

The scalar triple product $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 1 - 7k$ measures (up to a scale) the signed gap between the lines. For $k \neq \frac{1}{7}$ it is non-zero, so L_1 and $L_2(k)$ are skew with a positive shortest distance. As k increases through $\frac{1}{7}$ this quantity passes through 0: the lines momentarily become coplanar and intersect, then separate again on the other side. The shortest distance, as a function of k , falls to 0 at $k = \frac{1}{7}$ and is positive on either side.

R1R1

$k = \frac{1}{7}$ is the single value where the skew lines coincide in a plane and intersect; elsewhere they are skew.

A1

Examiner note / generalisation. The scalar triple product being zero is the universal coplanarity test for two lines; combined with “not parallel” it certifies an intersection.

Total: 28 marks (a)4 (b)3 (c)5 (d)5 (e)4 (f)4 (g)3

How this connects to the investigation: classifying the lines (a) motivates measuring their separation (b)–(d); technology confirms the exact result (e); then the parameter k turns the static picture into a family whose geometry changes at one critical value (f)–(g).