IB Demystified

Examiners | Moderators | Mentors

Mathematics: Analysis and Approaches

Higher Level

Paper 3 — Mock Examination

Mock Examination 3

Calculator:	A graphic display calculator is **required**
Time allowed:	1 hour 15 minutes
Maximum mark:	55 marks

Information for candidates

Please read the following before you begin.

- A graphic display calculator is required throughout this paper.
- You may use a clean copy of the *Mathematics: Analysis and Approaches HL* formula booklet.
- Answer **both** questions. Begin each question on a fresh page of your answer booklet.
- Unless a question directs otherwise, give numerical answers either exactly or rounded to three significant figures.
- Any result obtained from your calculator must be accompanied by suitable supporting working. Where a graph is used to reach an answer, include a sketch or a clear description of the method.
- Answers given without sufficient working may not receive full marks.
- The total available for this paper is **55 marks**. The time allowed is **1 hour 15 minutes**.

Candidate name: ....

Session number: ....

Date: ....

This paper consists of two investigative questions. *Do not turn over until you are ready to begin.*

1. [Maximum mark: 27]

This question compares two ways of modelling the growth of a population: an unrestricted (exponential) model and a model that includes a limit on resources (logistic). You will solve each, compare their predictions, and decide which is the more realistic.

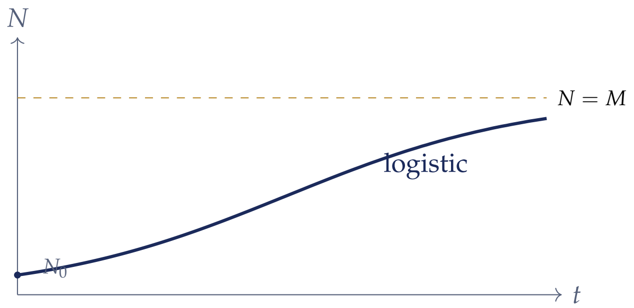
A fish population in a lake is measured in thousands and denoted $N(t)$, where t is the time in years. Initially $N(0) = 0.5$.

The **exponential model** assumes

$$\frac{dN}{dt} = kN, \qquad k > 0.$$

The **logistic model** assumes

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{M}\right), \qquad r = 0.4, \quad M = 5.$$



Illustrative logistic growth curve (axes not to a numerical scale).

(a) Verify that $N(t) = 0.5\,e^{kt}$ satisfies the exponential model. Given that this model predicts $N = 1.5$ when $t = 3$, find the exact value of k. **[3]**

(b) For the logistic model, find the equilibrium values of N (the values for which $\frac{dN}{dt} = 0$), and state which one is stable. **[3]**

(c) By separating the variables, show that the logistic model has solution

$$N(t) = \frac{M}{1 + A\,e^{-rt}},$$

where A is a constant. **[6]**

(d) Use the initial condition to find the value of A. **[2]**

(e) Using your calculator, find the time at which the logistic population first reaches 4 (thousand). **[3]**

(f) Find the population predicted by each model at $t = 10$, and comment on the comparison. **[4]**

(g) Determine the limiting value of N as $t \to \infty$ for each model, and interpret each result in context. **[3]**

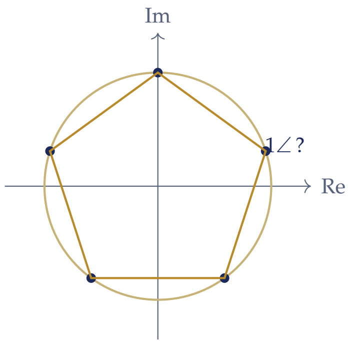
(h) Show that, for the logistic model, the growth rate $\frac{dN}{dt}$ is greatest when $N = \frac{M}{2}$, and find the time at which this occurs. **[3]**

This question is worth 27 marks. Question 2 begins on the next page.

2. **[Maximum mark: 28]**

This question investigates the complex roots of $z^n = 1$. After locating them as the vertices of a regular polygon, you will prove a striking product formula for the distances between vertices and use it to obtain an exact trigonometric value.

The solutions of $z^n = 1$ (where $n \geq 2$, $n \in \mathbb{Z}^+$) are called the nth roots of unity. Write $\omega = e^{2\pi i/n}$.



The fifth roots of unity (the exact positions are found in part (a)).

(a) Take $n = 5$. Write down the five fifth roots of unity in the form $e^{i\theta}$, and explain why they form the vertices of a regular pentagon inscribed in the unit circle. **[4]**

(b) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. **[3]**

(c) Generalise: prove that the sum of all n of the nth roots of unity is 0 for every $n \geq 2$. **[3]**

(d) Using the factorisation of $z^n - 1$, prove that

$$\prod_{k=1}^{n-1} \left| 1 - \omega^k \right| = n.$$

(The left-hand side is the product of the distances from the vertex at $z = 1$ to all the other vertices.) **[6]**

(e) For $n = 5$, use your calculator to verify the product formula in part (d). **[3]**

(f) Show that $\left| 1 - \omega^k \right| = 2 \sin \dfrac{k\pi}{n}$, and hence deduce that

$$\prod_{k=1}^{n-1} 2 \sin \frac{k\pi}{n} = n.$$

[4]

(g) Using part (f) with $n = 5$, find the exact value of $\sin \dfrac{\pi}{5} \sin \dfrac{2\pi}{5}$, and confirm it with your calculator. **[5]**

End of Mock Examination 3.